Some results on uniqueness of homogeneous differential polynomials of meromorphic functions

Dilip Chandra Pramanik*
Department of Mathematics
University of North Bengal
Raja Rammohunpur, Darjeeling-734013
West Bengal, India
dcpramanik.nbu2012@gmail.com

Jayanta Roy
Department of Mathematics, DDE
University of North Bengal
Raja Rammohunpur, Darjeeling-734013
West Bengal, India
jayantaroy983269@yahoo.com

Abstract. In this paper, we deal with the uniqueness problem of homogeneous differential polynomials of meromorphic functions sharing one small function $a(z)$ with weight l, where l is a non negative integer, with certain essential conditions and prove some results which generalize some results due to Lahiri and Pal [9].

Keywords: meromorphic function, differential polynomial, weighted sharing, small function, uniqueness.

1. Introduction and main results

Let f be a non-constant meromorphic function defined on the open complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f), m(r,f), N(r,f)$ (see, [2, 16, 21]). By $S(r,f)$ we denote any quantity satisfying the condition $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function a is called a small function with respect to f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

*. Corresponding author

are respectively called the deficiency and ramification index of a for the function f.

For any two nonconstant meromorphic functions f and g, and $a \in S(f) \cap S(g)$ we say that f and g share a IM (CM) provided that $f-a$ and $g-a$ have the same zeros ignoring (counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (CM), we say that f and g share ∞ IM (CM). Let f and g share 1 IM and let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q. By $\overline{N}_L(r,1;f)$ we denote the reduced counting function of those 1-points of f and g where $p > q \geq 1$; $\overline{N}_L(r,1;g)$ is defined similarly.

In this paper, we use n to denote any nonnegative integer. Also $\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$ is the order of f.

In 1976, M. Ozawa [12] proved the following theorem.

Theorem 1.1. *Let f and g be two non-constant entire functions such that f, g share the value 1 CM. If $\delta(0,f) > 0$ and 0 is a Picard exceptional value of g, then either $f \equiv g$ or $f.g \equiv 1$.*

Now, a days the problem on sharing values between two nonconstant meromorphic functions f and g and their relationship is an interesting area of research of uniqueness theory of functions.

In 1976, C. C. Yang [15] posted the following question:

Question 1.1. *Suppose that f and g are two transcendental entire functions such that f and g share the value 0 CM and $f^{(1)}$, $g^{(1)}$ share the value 1 CM. What can be said about the relationship between f and g ?*

Many authors, including Shibazaki [13], Yi [17, 18], Yang and Yi [19], Hua [3], Mues and Reinders [11], Lahiri [5, 6] studied the question.

In [18] H. X. Yi gave an answer to the above question of C. C. Yang [15] and proved the following result:

Theorem 1.2 ([18]). *Let f and g be two nonconstant meromorphic functions. If f, g share the value 0 CM, $f^{(n)}$, $g^{(n)}$ share the value 1 CM and $2\delta(0,f) + (n+2)\Theta(\infty,f) > n+3$, then either $f \equiv g$ or $f^{(n)}.g^{(n)} \equiv 1$.*

Later Yi [20] proved the following improvement of Theorem 1.2:

Theorem 1.3 ([20]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 CM and f, g share the value ∞ CM. If*

$$N(r,0;f) + N(r,0;g) + (n+2)\overline{N}(r,f) < (\lambda + o(1))T(r),$$

for $r \in I$, a set of infinite linear measure and λ is a positive constant < 1, then $f^{(n)}.g^{(n)} \equiv 1$ unless $f \equiv g$.

In 1997, Yi [22] proved some results which improved Theorems 1.2-1.3.

Theorem 1.4 ([22]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 CM. If*

$$2\delta(0,f)+(n+4)\Theta(\infty,f)>n+5 \text{ and}$$

$$2\delta(0,;g)+(n+4)\Theta(\infty,g)>n+5,$$

then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

Theorem 1.5 ([22]). *Let f and g be two nonconstant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 IM. If*

$$5\delta(0,f)+(4n+7)\Theta(\infty,f)>4n+11 \text{ and}$$

$$5\delta(0,g)+(4n+7)\Theta(\infty,g)>4n+11,$$

then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

In [10] Li and Li considered the problem of replacing the derivatives by linear differential polynomials. Let f be a nonconstant meromorphic function. An expression of the form

$$L(f)=f^{(n)}+a_{k-1}f^{(n-1)}+....+a_0 f,$$

where $a_0,a_1,...,a_{n-1}$ are complex constants, is called a linear differential polynomial generated by f.

Li and Li [10] proved that following theorems:

Theorem 1.6 ([10]). *Let f and g be two nonconstant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 CM and $\delta(0,f)>\frac{1}{2}$. If $\rho(f)\neq 1$, then either $f\equiv g$ or $L(f).L(g)\equiv 1$.*

Theorem 1.7 ([10]). *Let f and g be two nonconstant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 IM and $\delta(0,f)>\frac{4}{5}$. If $\rho(f)\neq 1$, then either $f\equiv g$ or $L(f).L(g)\equiv 1$.*

Definition 1.1. *Let m (≥ 1) be a positive integer, t (≥ 0) be an integer and let f be a nonconstant meromorphic function. An expression of the form*

$$P[f]=\sum_{k=1}^{m}a_k\prod_{j=0}^{t}\left(f^{(j)}\right)^{l_{kj}}, \tag{1}$$

where $a_k\in S(f)$ for $k=1,2,......,m$ and l_{kj} $(1\leq k\leq m;\ 0\leq j\leq t)$ are nonnegative integers and $d=\sum_{j=0}^{t}l_{kj}$ for $k=1,2,......,m$, is called a homogeneous differential polynomial of degree d generated by f. Also we denote by Q the quantity $Q=\max_{1\leq k\leq m}\sum_{j=0}^{t}jl_{kj}$.

Let f and g be two nonconstant meromorphic functions. When we consider $P[f]$ and $P[g]$ are nonconstant homogeneous differential polynomials of f and g respectively, then we understand that the coefficients $a_j \in S(f) \cap S(g)$.

Recently in 2017 Lahiri and Pal [9] extended the results of Li and Li [10] to homogeneous differential polynomial and obtained the following theorem.

Theorem 1.8. *Let f and g be two nonconstant meromorphic functions, a ($\not\equiv 0, \infty$) $\in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, as defined by (1) are nonconstant. If $P[f]$ and $P[g]$ share a IM and*

$$\min\left\{5\delta(0,f) + \frac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g) + \frac{4Q+7}{d}\Theta(\infty,g)\right\}$$
$$> \frac{4Q+4d+7}{d}, \tag{2}$$

then either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

Remark 1.1. If $P[f]$ and $P[g]$ share a CM, then the condition (2) of Theorem 1.8 can be replaced by the following

$$\min\left\{2\delta(0,f) + \frac{Q+4}{d}\Theta(\infty,f), 2\delta(0,g) + \frac{Q+4}{d}\Theta(\infty,g)\right\} > \frac{Q+d+4}{d}.$$

So, one may ask the following question which is the motivation of this paper.

Question 1.2. *Keeping the conclusion of the above theorem intact is it possible to relax the nature of sharing the function a between CM and IM ?*

To investigate this problem we use a new notion of scaling between CM and IM known as weighted sharing, which is introduced in [7, 8]. In the following definition we explain this notion.

Definition 1.2 ([7, 8])**.** *Let l be a nonnegative integer or infinity and $a \in S(f)$. We denote by $E_l(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity m is counted m times if $m \le l$ and $l+1$ times if $m > l$. If $E_l(a,f) = E_l(a,g)$, we say that f, g share the function a with weight l. We write f and g share (a,l) to mean that f and g share the function a with weight l. Since $E_l(a,f) = E_l(a,g)$ implies that $E_s(a,f) = E_s(a,g)$ for any integer s $(0 \le s < l)$, if f, g share (a,l), then f, g share (a,s), $(0 \le s < l)$. Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a,0)$ or (a,∞) respectively.*

In this paper, we try to give a positive answer to question 1.2 and obtain the following results.

Theorem 1.9. *Let f and g be two nonconstant meromorphic functions, a ($\not\equiv 0, \infty$) $\in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, as defined by (1) are nonconstant. If $P[f]$ and $P[g]$ share (a,l) with one of the following conditions:*

(i) $l = \infty$ *and*

$$\min\left\{2\delta(0,f)+\frac{Q+4}{d}\Theta(\infty,f),2\delta(0,g)+\frac{Q+4}{d}\Theta(\infty,g)\right\}>\frac{Q+d+4}{d}, \tag{3}$$

(ii) $0 < l < \infty$ *and*

$$\begin{aligned}\min\Big\{&\frac{(2l+1)d}{l}\delta(0,f)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,f),\\&\frac{(2l+1)d}{l}\delta(0,g)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,g)\Big\}\\&>\frac{(l+1)d+Q+1}{l}+4+Q,\end{aligned} \tag{4}$$

(iii) $l = 0$ *and*

$$\min\left\{5\delta(0,f)+\frac{4Q+7}{d}\Theta(\infty,f),5\delta(0,g)+\frac{4Q+7}{d}\Theta(\infty,g)\right\}>\frac{4Q+4d+7}{d}, \tag{5}$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Theorem 1.10. *Let* f *and* g *be two nonconstant meromorphic functions,* a $(\not\equiv 0,\infty) \in S(f)\cap S(g)$. *Suppose* $P[f]$ *and* $P[g]$, *as defined by (1) are nonconstant. If* f *and* g *share the value* 0 *CM and* ∞ *IM and* $P[f]$, $P[g]$ *share* (a,l) *with one of the following conditions:*

(i) $l = \infty$ *and*

$$2\delta(0,f)+\frac{Q+4}{d}\Theta(\infty,f)>\frac{Q+d+4}{d},$$

(ii) $0 < l < \infty$ *and*

$$\frac{(2l+1)d}{l}\delta(0,f)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,f)>\frac{(l+1)d+Q+1}{l}+4+Q,$$

(iii) $l = 0$ *and*

$$5\delta(0,f)+\frac{4Q+7}{d}\Theta(\infty,f)>\frac{4Q+4d+7}{d},$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Theorem 1.11. *Let* f *and* g *be two nonconstant entire functions,* a $(\not\equiv 0,\infty) \in S(f)\cap S(g)$. *Suppose* $P[f]$ *and* $P[g]$ *are nonconstant homogeneous differential polynomials of degree* d *as defined by (1). If* f *and* g *share the value* 0 *CM,* $P[f]$ *and* $P[g]$ *share* (a,l) *with one of the following conditions:*

(i) $l = \infty$ *and*

$$\delta(0,f)>\frac{1}{2},$$

(ii) $0 < l < \infty$ *and*

$$\delta(0, f) > \frac{l+1}{2l+1},$$

(iii) $l = 0$ *and*

$$\delta(0, f) > \frac{4}{5},$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Corollary 1.1. *Let f and g be two nonconstant entire functions such that $L(f)$ and $L(g)$ are nonconstant linear differential polynomials. If f and g share the value 0 CM, $L(f)$ and $L(g)$ share $(1, l)$ with one of following conditions:*
(i) $l = \infty$ *and*

$$\delta(0, f) > \frac{1}{2},$$

(ii) $0 < l < \infty$ *and*

$$\delta(0, f) > \frac{l+1}{2l+1},$$

(iii) $l = 0$ *and*

$$\delta(0, f) > \frac{4}{5},$$

then either $f = g$ or $L(f).L(g) \equiv 1$ under any one of the following conditions:
(i) $\rho(f) \neq 1$,
(ii) $\rho(f) = 1$ *and (a) f has at most a finite number of zeros, or*
(b) f has infinitely many zeros and f is of minimal type.

Suppose F and G share $(1, l)$ and let z_0 be a zero of $F - 1$ of multiplicity p and a zero of $G - 1$ of multiplicity q. We define by $\overline{N}^{L}_{(l+1}(r, 1; F)$ the reduced counting function of those 1-points of F and G where $p > q \geq l+1$; $\overline{N}^{L}_{(l+1}(r, 1; G)$ is defined similarly. Note that for $l = 0$, we have $\overline{N}^{L}_{(l+1}(r, 1; F) = \overline{N}_L(r, 1; F)$. Also denote by $N^{1)}_{E}(r, 1; F)$ the counting function of those 1-points of F and G where $p = q = 1$ and by $\overline{N}^{(2}_{E}(r, 1; F)$ the counting function of those 1-points of F and G where $p = q \geq 2$, where each such zero is counted only once.

2. Lemmas

Let F and G be two nonconstant meromorphic functions. We shall define by H the following function

$$H = \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right).$$

Lemma 2.1 ([9])**.** *Let f be a nonconstant meromorphic function and $P[f]$ be defined as (1). Then*

$$\begin{aligned} (i)T(r,P) &\leq dT(r,f)+Q\overline{N}(r,\infty;f)+S(r,f). \\ (ii)N(r,0;P) &\leq T(r,P)-dT(r,f)+dN(r,0;f)+S(r,f) \\ &\leq Q\overline{N}(r,\infty;f)+dN(r,0;f)+S(r,f). \end{aligned}$$

Lemma 2.2 ([1])**.** *Let F and G be two nonconstant meromorphic functions sharing $(1,l)$ and $H\not\equiv 0$. Then*

$$\begin{aligned} N(r,\infty,H) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;G) \\ &+ \overline{N}_{(2}(r,0;F)+\overline{N}_L(r,1;F) \\ &+ \overline{N}_L(r,1;G)+\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G). \end{aligned}$$

Lemma 2.3 ([4])**.** *Let f be a transcendental meromorphic function and $P[f]$ be a homogeneous differential polynomial generated by f of degree $d\geq 1$. Then*

$$dT(r,f)\leq \overline{N}(r,\infty;f)+\overline{N}(r,1;F)+N(r,0;f^d)-N_0(r,0;(P(f))^{(1)})+S(r,f),$$

where $N_0(r,0;(P[f])^{(1)})$ denotes the counting function corresponding to the zeros of $(P[f])^{(1)}$ which are not the zeros of $P[f]$ and $P[f]-1$.

Lemma 2.4 ([14])**.** *Let f be a nonconstant meromorphic function and let*

$$p(f)=a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0,$$

where $a_i\in S(f)$ for $i=0,\ 1,...\ n,\ a_n\ (\neq 0)$ be a polynomial of degree n. Then $T(r,p(f))=nT(r,f)+S(r,f)$.

Lemma 2.5. *If F and G be two nonconstant meromorphic functions sharing $(1,l)$ where l is positive integer and $H\not\equiv 0$. Then*

$$\begin{aligned} T(r,F) &\leq (1+\frac{1}{l})N(r,0;F)+(2+\frac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)+2\overline{N}(r,\infty;G) \\ &+ S(r,F)+S(r,G). \end{aligned}$$

Proof. Suppose $H\not\equiv 0$. Then by a simple calculation we see that

$$\begin{aligned} N_E^{1)}(r,1;F) &\leq N(r,0;H)\leq T(r,H)+O(1) \\ &\leq N(r,\infty;H)+S(r,F)+S(r,G). \end{aligned} \tag{6}$$

Since F and G share $(1,l)$, we have from Lemma 2.2

$$\begin{aligned} N(r,\infty;H) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}^L_{(l+1}(r,1;F) \\ &+\overline{N}^L_{(l+1}(r,1;G)+\overline{N}_{(2}(r,0;F) \\ &+\overline{N}_{(2}(r,0;G)+\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G). \end{aligned} \tag{7}$$

By Nevanlinna's Second Fundamental Theorem we have

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+\overline{N}(r,0;G) \\ &+ \overline{N}(r,\infty;G)+\overline{N}(r,1;G)-N_0(r,0;F^{(1)}) \\ &- N_0(r,0;G^{(1)})+S(r,F)+S(r,G), \end{aligned} \tag{8}$$

where $N_0(r,0;F^{(1)})$ denotes the counting function corresponding to the zeros of $F^{(1)}$ which are not the zeros of F and $F-1$. Similarly we define $N_0(r,0;G^{(1)})$.

Since F and G share $(1,l)$, using (6) and (7) we have

$$\begin{aligned} &\overline{N}(r,1;F)+\overline{N}(r,1;G) \\ &= N_E^{1)}(r,1;F)+\overline{N}_{(l+1}^L(r,1;F)+\overline{N}_{(l+1}^L(r,1;G) \\ &+ N_E^{(2}(r,1;F)+\overline{N}(r,1;G) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G) \\ &+ 2\overline{N}_{(l+1}^L(r,1;F)+2\overline{N}_{(l+1}^L(r,1;G)+N_E^{(2}(r,1;F)+\overline{N}(r,1;G) \\ &+ \overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G) \\ &+ \overline{N}_{(l+1}^L(r,1;F)+\overline{N}_{(l+1}^L(r,1;G)+N(r,1;G)+\overline{N}_0(r,0;F^{(1)}) \\ &+ \overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G) \\ &+ \overline{N}_{(l+1}^L(r,1;F)+\overline{N}_{(l+1}^L(r,1;G)+T(r,G)+\overline{N}_0(r,0;F^{(1)}) \\ &+ \overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G). \end{aligned} \tag{9}$$

Also, we have

$$\begin{aligned} \overline{N}_{(l+1}^L(r,1;F)+\overline{N}_{(l+1}^L(r,1;G) &\leq \frac{1}{l}N(r,0;F^{(1)})+S(r,F) \\ &\leq \frac{1}{l}N(r,0;F)+\frac{1}{l}\overline{N}(r,\infty;F)+S(r,F). \end{aligned} \tag{10}$$

Now, using (9), (10) in (8) we get

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F) \\ &+ \overline{N}(r,0;G)+\overline{N}_{(2}(r,0;G)+\frac{1}{l}N(r,0;F)+\frac{1}{l}\overline{N}(r,\infty;F) \\ &+ T(r,G)+S(r,F)+S(r,G). \end{aligned}$$

Since $\overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F)\leq N(r,0;F)$, we have

$$\begin{aligned} T(r,F) &\leq (1+\frac{1}{l})N(r,0;F)+(2+\frac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)+2\overline{N}(r,\infty;G) \\ &+ S(r,F)+S(r,G). \end{aligned}$$

□

3. Proof of main theorems

Proof of Theorem 1.9:

Proof. Let

$$F=\frac{P[f]}{a},\ G=\frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share (a,l), it follows that F, G share $(1,l)$ except at the zeros and poles of a.

Now, we consider the following cases:
Case 1: $l=\infty$. This case follows from Remark 1.1.
Case 2: $0<l<\infty$.

Suppose $H\not\equiv 0$. From Lemma 2.5 and Lemma 2.1, we get

$$\begin{aligned} T(r,F) &\leq (1+\frac{1}{l})N(r,0;F)+(2+\frac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)\\ &\quad +2\overline{N}(r,\infty;G)+S(r,F)+S(r,G).\\ &\quad \Rightarrow dT(r,f)\leq \frac{Q+1+2l}{l}\overline{N}(r,\infty;f)+(2+Q)\overline{N}(r,\infty;g)\\ &\quad +\frac{(l+1)d}{l}N(r,0;f)+dN(r,0;g)+S(r,f)+S(r,g). \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} dT(r,g) &\leq \frac{Q+1+2l}{l}\overline{N}(r,\infty;g)+(2+Q)\overline{N}(r,\infty;f)+\frac{(l+1)d}{l}N(r,0;g)\\ &+ dN(r,0;f)+S(r,f)+S(r,g). \end{aligned} \tag{12}$$

Combining (11) and (12), we get

$$\begin{aligned} dT(r,f)+dT(r,g) &\leq (\frac{Q+1+2l}{l}+Q+2)\overline{N}(r,\infty;f)+\frac{(2l+1)d}{l}N(r,0;f)\\ &+ (\frac{Q+1+2l}{l}+Q+2)\overline{N}(r,\infty;g)+\frac{(2l+1)d}{l}N(r,0;g)\\ &+ S(r,f)+S(r,g). \end{aligned}$$

$$\begin{aligned} &\left\{\frac{(2l+1)d}{l}\delta(0,f)+(\frac{Q+1+2l}{l}+Q+2)\Theta(\infty,f)\right.\\ &\left.-\frac{(l+1)d+Q+1}{l}-4-Q\right\}T(r,f)\\ &+\left\{\frac{(2l+1)d}{l}\delta(0,g)+(\frac{Q+1+2l}{l}+Q+2)\Theta(\infty,g)\right.\\ &\left.-\frac{(l+1)d+Q+1}{l}-4-Q\right\}T(r,g)\\ &\leq S(r,f)+S(r,g), \end{aligned}$$

which contradict (4). Therefore $H \equiv 0$ and so integrating twice we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where $A\ (\neq 0)$ and B are constants.

Thus,

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)} \tag{13}$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}. \tag{14}$$

Next, we consider following three subcases:

Subcase 2.1: $B \neq 0, -1$. Then, from (14) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r, \infty; F).$$

By Nevanlinna Second Fundamental Theorem and (ii) of Lemma 2.1 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{B+1}{B}; G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + dN(r,0;g) + \overline{N}(r,\infty;F) + S(r,G), \end{aligned}$$

$$\Rightarrow dT(r,g) \leq \overline{N}(r,\infty;f) + dN(r,0;g) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \tag{15}$$

If $A - B - 1 \neq 0$, then it follows from (13) that

$$\overline{N}(r, \frac{-A+B+1}{B+1}; F) = \overline{N}(r,0;G).$$

Using Nevanlinna Second Fundamental Theorem and Lemma 2.1

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r, \frac{-A+B+1}{B+1}; F) + S(r,F),$$

$$\begin{aligned} &\Rightarrow dT(r,f) \leq \overline{N}(r,\infty;f) + dN(r,0;f) \\ &+ Q\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g). \end{aligned} \tag{16}$$

Adding (15) and (16), we obtain

$$\begin{aligned} T(r,f) + T(r,g) &\leq N(r,0;f) + \frac{2}{d}\overline{N}(r,\infty;f) + 2N(r,0;g) \\ &+ \frac{Q+1}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g), \end{aligned}$$

which contradicts (4).

Therefore $A-B-1=0$. Then from (13), it follows that

$$\overline{N}(r,-\frac{1}{B};F)=\overline{N}(r,\infty;G).$$

Again by Nevanlinna Second Fundamental Theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,-\frac{1}{B};F)+S(r,F) \\ &\leq \overline{N}(r,\infty;f)+T(r,F)-dT(r,f)+dN(r,0;f) \\ &\quad +\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \end{aligned}$$

$$(17) \quad \Rightarrow dT(r,f)\leq \overline{N}(r,\infty;f)+dN(r,0;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g).$$

Combining (15) and (17), we get

$$\begin{aligned} T(r,f)+T(r,g) &\leq N(r,0;f)+\frac{2}{d}\overline{N}(r,\infty;f)+N(r,0;g) \\ &+ \frac{2}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g), \end{aligned}$$

which violates (4).

Subcase 2.2: $B=-1$. Then from (13) and (14), we get

$$G=\frac{A}{A+1-F}$$

and

$$F=\frac{(1+A)G-A}{G}.$$

If $A+1\neq 0$, then

$$\overline{N}(r,A+1;F)=\overline{N}(r,\infty;G),$$

$$\overline{N}(r,\frac{A}{A+1};G)=\overline{N}(r,0;F).$$

By similar argument as in Subcase 2.1 we arrive at a contradiction.

Therefore, $A+1=0$, then

$$FG=1$$

$$\Rightarrow P[f].P[g]\equiv a^2.$$

Subcase 2.3: $B=0$. Then (13) and (14) gives $G=\frac{F+A-1}{A}$ and $F=AG+1-A$. If $A-1\neq 0$, $\overline{N}(r,1-A;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,\frac{A-1}{A};G)=\overline{N}(r,0;F)$. Proceeding similarly as in Subcase 2.1 we get a contradiction.

Therefore $A-1=0$ then $F\equiv G$ i.e.,

$$P[f]\equiv P[g].$$

Case 3: $l=0$. The conclusion follows from Theorem 1.1 of [9].

This complete the proof. □

Proof of Theorem 1.10:

Proof. Let

$$F = \frac{P[f]}{a}, \ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share (a, l), it follows that F, G share $(1, l)$ except at the zeros and poles of a. By Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned} dT(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,1;F) + N(r,0;f^d) + S(r,f) \\ &= \overline{N}(r,\infty;g) + \overline{N}(r,1;G) + N(r,0;g^d) + S(r,f) \\ &\leq (1+Q+2d)T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{18}$$

Similarly,

$$dT(r,g) \leq (1+Q+2d)T(r,f) + S(r,f) + S(r,g). \tag{19}$$

From (18) and (19) we get $S(r,f) = S(r,g)$. The rest of the proof is similar to that of Theorem 1.9. □

Proof of the Corollary 1.1:

Proof. By Theorem 1.11 we get either $L(f) \equiv L(g)$ or $L(f).L(g) \equiv 1$. Let $L(f) \equiv L(g)$ so that $L(f-g) \equiv 0$. Proceeding similarly as in the proof of Corollary 1.2 of [9], we obtain $f = g$. □

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